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Differential Operator Method for One-Dimensional Inverse Scattering Problems

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Abstract. The differential operator method for solving the Gel'fand-Levitan integral equation is generalized, which transforms the original integral equation into a partial differential equation with boundary conditions. A specific case for obtaining analytical solutions is given.

INTRODUCTION

Consider one dimensional scattering from the surface $z = 0$ of a stratified inhomogeneous region $z > 0$. When a plane wave projects normally onto this surface, a reflected wave will be returned to the homogeneous region $z < 0$, and the total field in region $z < 0$ can be written as

$$E(k,z) = e^{jkz} + r(k)\,e^{-jkz}, \qquad (z < 0), \tag{1}$$

where k is the wave-number, $r(k)$ is the reflection coefficient. It is well known that a one dimensional wave equation in the region $z > 0$ can be transformed into the one dimensional Schrödinger equation:

$$\frac{d^2E(k,z)}{dz^2} + [k^2 - V(z)]\,E(k,z) = 0, \qquad (z > 0), \tag{2}$$

where $V(z)$ is a real potential function which describes the properties of the inhomogeneous region.

One-dimensional inverse scattering problems involve the reconstruction of $V(z)$ from the known $r(k)$. The governing equation of one dimensional inverse scattering problems is the well known Gel'fand-Levitan integral equation [1]:

$$R(z+y) + K(z,y) + \int_{-y}^{z} K(z,z')\,R(y+z')\,dz' = 0, \qquad (-z \le y \le z,\ \ z > 0), \tag{3}$$

$$V(z) = 2\,\frac{dK(z,z)}{dz}, \tag{4}$$

where $R(z)$ is the Fourier transform of $r(k)$:

$$R(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)\,e^{-jkz}\,dk - j\sum_{\nu=1}^{N} r_\nu\, e^{-jk_\nu z}, \tag{5}$$

where N is the number of poles of $r(k)$, $\{k_\nu\}$ are the positions of poles on the lower half k-plane, and r_ν is the residual of $r(k)$ at k_ν. Several methods for solving the G-L equation have been published [2–4]. In this paper, the differential operator method for a rational function $r(k)$ is generalized, which transforms the G-L integral equation into a partial differential equation with boundary conditions. A specific case of a known reflection coefficient $r(k)$ is considered for obtaining analytical solutions for $K(z,z)$ and then $V(z)$.

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THE DIFFERENTIAL OPERATOR METHOD

The basic idea of the differential operator method is finding a function $f(p)$ of the differential operator $p = \frac{\partial}{\partial y}$ which satisfies the condition:

$$f(p)\, R(z+y) = 0, \tag{6}$$

and then transforming the G-L integral equation (3) into a differential equation for the unknown function $K(z,y)$ by using $f(p)$. Usually, a polynomial $f(p)$ is easy to construct, as is considered here.

To show how the method works, we consider the practical model of a reflection coefficient $r(k)$ which corresponds to the reflection from electron-layers [3–4]:

$$r(k) = \frac{N(k)}{\prod_{i-1}^{n}(k-k_i)}, \tag{7}$$

where $N(k)$ is a polynomial of degree $n-2$ or less; all k_i are simple poles located on the lower half k-plane.

The Fourier transform of (7) is:

$$R(\xi) = \begin{cases} -j\sum_{i=1}^{n} \alpha_i\, e^{-jk_i\xi}, & \xi > 0, \\ 0, & \xi < 0, \end{cases} \tag{8}$$

with

$$\alpha_i = \frac{N(k_i)}{\prod_{l=1,l\neq i}^{n} (k_l - k_i)}, \qquad (i = 1, 2, \ldots, n). \tag{9}$$

Following the basic idea, we now construct a polynomial

$$f(p) = \sum_{l=0}^{m} C_l\, p^l = \sum_{l=0}^{m} C_l\, \frac{\partial^l}{\partial y^l}; \tag{10}$$

the constants $\{C_l\}$ can be determined by using condition (6):

$$\sum_{l=0}^{m} C_l\, \frac{\partial^l}{\partial \xi^l} R(\xi) = 0. \tag{11}$$

Substituting (8) into (11), and interchanging the order of summations, we get

$$\sum_{i=1}^{n} \alpha_i\, e^{-jk\xi} \left[\sum_{l=0}^{m} C_l\, (-jk_i)^l\right] = 0, \tag{12}$$

which should exist for any ξ, so that a system of n linear algebraic equations for the $m+1$ coefficients $\{C_l\}$ is obtained:

$$\sum_{l=0}^{m} C_l\, (-jk_i)^l = 0, \qquad (i = 1, 2, \ldots, n). \tag{13}$$

Upon setting $m = n$, and $C_n = 1$ for normalizing the solutions of $\{C_l\}$, we obtain the solution of (13):

$$\begin{cases} C_{n-1} = j\sum_{i=1}^{n} k_i, & \\ C_{n-2} = -\sum_{i=1}^{n}\sum_{j=1}^{n} k_i\, k_j, & (i \neq j), \\ C_{n-3} = -j\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} k_i\, k_j\, k_m, & (i \neq j,\ j \neq m,\ m \neq i), \\ \cdots & \\ C_0 = j^n\, k_1\, k_2 \ldots k_n. & \end{cases} \tag{14}$$

Therefore

$$f(p) = p^n + j\Big(\sum_{i=1}^{n} k_i\Big)p^{n-1} + \cdots + j^n\, k_1\, k_2 \ldots k_n, \tag{15}$$

which can be factorized as:

$$f(p) = (p + jk_1)\,(p + jk_2)\ldots(p + jk_n). \tag{16}$$

To transform the G-L integral equation (3) into a differential equation, we apply this constructed $f(p)$ to (3) and, taking care of the relation

$$\begin{aligned} p^n \int_{-y}^{z} K(z,z')\,R(y+z')\,dz' = \int_{-y}^{z} & K(z,z')\,R^{(n)}(y+z')dz' \\ & + K(z,-y)\,R^{(n-1)}(0) + R^{(n-2)}(0)\,\frac{\partial K(z,-y)}{\partial y} \\ & + \cdots + R'(0)\,\frac{\partial^{(n-2)}K(z,-y)}{\partial y^{(n-2)}} + R(0)\,\frac{\partial^{(n-1)}K(z,-y)}{\partial y^{(n-1)}}, \end{aligned} \tag{17}$$

a differential equation is obtained:

$$f(p)\,K(z,y) + [A_1\,p^{n-1} + A_2\,p^{n-2} + \cdots + A_{n-1}\,p + A_n]\,K(z,-y) = 0, \tag{18}$$

where

$$\begin{cases} A_1 = R(0), \\ A_2 = R(0) + R'(0), \\ \quad\cdots \\ A_n = R(0) + R'(0) + \cdots + R^{(n-1)}(0). \end{cases} \tag{19}$$

Changing the sign of y in (18), we get a similar equation:

$$f(-p)\,K(z,-y) + [(-1)^{n-1}A_1\,p^{n-1} + (-1)^{n-2}A_2\,p^{n-2} + \cdots + A_n]\,K(z,y) = 0. \tag{20}$$

Applying $f(p)$ and $f(-p)$ to (20) and (18), respectively, then eliminating $K(z,-y)$, we obtain a differential equation with respect to $K(z,y)$ only:

$$[f(p)\,f(-p) - g_1(p)\cdot g_2(p)]\,K(z,y) = 0, \tag{21}$$

where

$$\begin{aligned} g_1(p) &= A_1\,p^{n-1} + A_2\,p^{n-2} + \cdots + A_n, \\ g_2(p) &= (-1)^{n-1}A_1\,p^{n-1} + (-1)^{n-2}A_2\,p^{n-2} + \cdots + A_n. \end{aligned}$$

Since

$$f(p)\,f(-p) = (-1)^n\,(p^2 + k_1^2)\,(p^2 + k_2^2)\ldots(p^2 + k_n^2), \tag{22}$$

equation (21) is a linear partial differential equation of order $2n$ with constant coefficients, which is easy to deal with because of its special form. After the general solution of (21) is found, $2n$ integral constants will appear and can be determined by $2n$ boundary conditions, which are derived by applying p^m to (3) at $y = -z$:

$$\begin{cases} R^{(m)}(0) + R^{(m-1)}(0)\,K(z,z) + R^{(m-2)}(0)\left[\dfrac{\partial K(z,-y)}{\partial y}\right]_{y=-z} \\ \qquad + \cdots + R'(0)\left[\dfrac{\partial^{(m-2)}K(z,-y)}{\partial y^{(m-2)}}\right]_{y=-z} + R(0)\left[\dfrac{\partial^{(m-1)}K(z,-y)}{\partial y^{(m-1)}}\right]_{y=-z} \\ \qquad + \left[\dfrac{\partial^m K(z,-y)}{\partial y^m}\right]_{y=-z} = 0, \qquad\qquad m = 0,1,2,\ldots,2n-1. \end{cases} \tag{23}$$

It is important to notice that, no matter what the form of $r(k)$ is, whenever a polynomial $f(p)$ is found, a differential equation (21) and boundary conditions (23) can be used to find $K(z,y)$ and then $V(z)$. In this case, equation (21) and boundary conditions (23) can be used to take the place of the G-L equation (3), the latter is not needed.

EXAMPLE OF ANALYTICAL SOLUTION

When the numerator in (7) is a constant, such as $N(k) = k_1 k_2 \dots k_n$, it is easy to prove that

$$R(0) = R'(0) = \cdots = R^{(n-2)}(0) = 0, \tag{24}$$

and then

$$A_1 = A_2 = \cdots = A_{n-1} = 0, \quad A_n = R^{(n-1)}(0); \tag{25}$$

the differential equation (21) is then reduced to

$$f(p)\, f(-p)\, K(z,y) - [R^{(n-1)}(0)]^2\, K(z,y) = 0. \tag{26}$$

In this case, analytical solutions exist for $n \leq 4$. In the cases of $n = 2$ and $n = 3$, we obtain the same results with [3] and [4] by using the above formulas. We also have some analytical solutions in the case of $n = 4$, which has not been considered before.

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